Efficient Generation of Body-Fitted Coordinates for Cascades Using Multigrid

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Abstract

A METHOD for the generation of body-fitted curvilinear coordinate systems is discussed and a technique using multigrid is proposed to reduce the computing time and to improve the accuracy. This technique is used to accelerate the convergence of point and line SOR relaxation schemes. The computing time is reduced, respectively, by factors of 2 and 3 over the usual point and line SOR. The accuracy for a given computational effort is improved by one and two orders of magnitude, respectively, when compared with point SOR. The multigrid scheme is then applied to the generation of curvilinear coordinates for turbine cascades.

Contents

The numerical computation of a flowfield requires an adequate treatment of the boundary conditions which can be quite difficult to incorporate for complex geometries encountered in practical engineering problems. This can be resolved by the use of an appropriate coordinate system where the coordinate lines coincide with the boundaries. A second characteristic is the ability to stretch the grid in order to concentrate more nodes in regions where high gradients of flow properties are expected. The main advantage is that both the mesh generation and the solution of the problem of interest are solved on a rectangular mesh. This lends itself to a very simple discretization using finite differences or finite volumes particularly at the boundaries where all interpolations are avoided. Another possible application is the automatic generation of a finite element mesh.

Transformations that map arbitrary physical regions into a rectangle with the above characteristics have been proposed by several authors[1,2] and are called body-fitted curvilinear coordinate systems. The approach consists of solving a system of elliptic equations which yield the physical coordinates in terms of the transformed coordinates. The physical coordinates $Z=Z(\tau,\eta)$ and $\phi=\phi(\tau,\eta)$ are obtained by solving the following system of nonlinear elliptic coupled equations

$$\alpha Z_{\eta\eta}+\gamma Z_{\tau\tau}-2\beta Z_{\eta\tau}+J^2QZ_\eta+J^2RZ_\tau=0$$

$$\alpha\phi_{\eta\eta}+\gamma\phi_{\tau\tau}-2\beta\phi_{\eta\tau}+J^2Q\phi_\eta+J^2R\phi_\tau=0 \qquad (1)$$

where

$$\alpha=\phi_\tau^2+Z_\tau^2,\ \gamma=\phi_\eta^2+Z_\eta^2,\ \beta=\phi_\eta\phi_\tau+Z_\eta Z_\tau,\ J=\phi_\eta Z_\tau-\phi_\tau Z_\eta$$

The terms Q and R are functions used to concentrate the coordinate lines in the physical domain. The boundary conditions for this system are simply the equations for the boundary shapes and are specified as a set of (Z,ϕ) values at a number of nodes on the boundary of the (τ,η) domain. This problem has been solved using a relaxation procedure[1] and with an ADI scheme after introducing an artificial time.[2] In this study a novel and efficient technique for accelerating the convergence is used and compared with the classical relaxation procedures.

Multigrid Method

The essential idea of the multigrid method lies in the discretization of the same problem on several grids of different mesh widths. One then solves successively on each of these a residual equation that is the original equation, to which is added a forcing term. This forcing term, it will be seen later, represents the difference of the discretization errors betwen the fine and the coarser grids. A complete multigrid cycle consists in relaxation sweeps on each of the coarser grids as well as on the fine grid. It is from the excursions on the coarse grids, together with the nature of the forcing term, that the method draws its efficiency, which can be explained as follows.

The effect of a relaxation sweep is to smooth the residual resulting from a given initial solution. This residual may be thought of as having several components corresponding to different wavelengths. Since the correction equation at a given node involves only the point and its immediate neighbors, then only those wavelengths of the order of the mesh width are effectively damped out. The other components persist much longer and this explains the typical behavior of a relaxation scheme whereby the residual initially drops rapidly and then levels off, and it is precisely this latter part of the calculation that is lengthy and thus expensive to obtain. By relaxing on different grids, one damps out different wavelengths corresponding to the grid meshes. Consequently, a multigrid cycle will liquidate simultaneously several wavelength components and provide a much more efficient scheme.

The derivation for the correction equation (one per grid) for the multigrid method will be outlined briefly.[3] The differential equations to be solved, Eqs. (1), are denoted for convenience by the operator L, where

$$L\equiv\alpha\frac{\partial^2}{\partial\eta^2}+\gamma\frac{\partial^2}{\partial\tau^2}-2\beta\frac{\partial^2}{\partial\eta\partial\tau}+J^2Q\frac{\partial}{\partial\eta}+J^2R\frac{\partial}{\partial\tau} \qquad (2)$$

This operator is discretized on a hierarchy of grids G^0, $G^1,\ldots,G^M$, using centered finite differences. An approximate numerical solution ϕ^M, Z^M is sought on G^M, the finest grid, such that

$$L^M\phi^M=0 \text{ and } L^MZ^M=0 \qquad (3)$$

This is carried out by relaxation of a starting solution. However, before this is done on the fine grid, the solution is "skimmed" by relaxing on the coarse and intermediate grids, G^K, the appropriate residual equation. This residual equation differs from the original equation by the presence of a forcing

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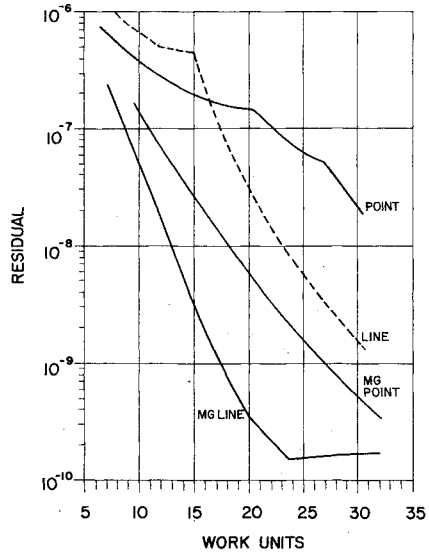

Fig. 1 Comparison of the relaxation histories of four methods for the generation of curvilinear coordinates.

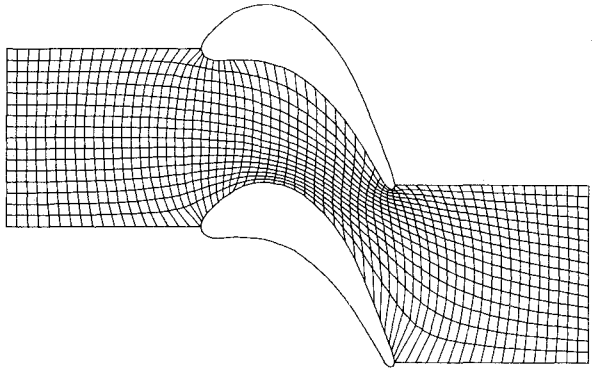

Fig. 2 Curvilinear coordinates for a typical cascade using the present technique on a 17×65 grid.

term,

$$L^K\phi^K = F^K \quad \text{and} \quad L^K Z^K = F^K \tag{4}$$

where the forcing term F^K is defined as

$$F^K = I_M^K L^M \phi^M - L^K I_M^K \phi^M \tag{5}$$

and I_M^K indicates the interpolation of a solution from G^M to G^K. The forcing term is interpreted as the difference of the discretization errors of the operator on the fine and the current grid.

When a sufficient accuracy is reached on a given grid, the solution is then interpolated to the next, where the appropriate residual equation is relaxed, and so on. It is noted that the same relaxation scheme is used throughout on all grids. Most of the computational effort is spent on the coarse grids where the computational cost is low, as they contain very few points.

Results

After numerous exploratory computations the effect of certain parameters was assessed.[4] The choice of an over-relaxation factor was found to have little effect on the relaxation history. By varying the number of grids, little advantage is obtained from more than four grids. Finally, the number of sweeps carried out on each grid was varied and it was found that a fixed strategy using three sweeps on each grid gave the best results. Using a smaller number destroyed the beneficial effects of multigrid, whereas a larger number, say four or five, simply increased computer time without significantly improving the convergence rate.

The multigrid algorithm was applied to accelerate the convergence of both point and line SOR. The results are shown in Fig. 1, which compares the relaxation histories of those methods with and without multigrid. (In the latter case optimum over-relaxation is shown.) These computations were carried out for the cascade of Fig. 2, which also shows the resulting grid with a 17×65 mesh.

The measure for the error used is the root-mean-square value of the residual, and the unit for computer time is chosen as the time required to complete an entire relaxation sweep, and is called a work unit.

Conclusions

Body-fitted coordinates can be generated more efficiently using the multigrid method than with either optimum point or line SOR. For the same computer time, multigrid will yield a residual one order of magnitude smaller than optimum line SOR and at least two orders of magnitude smaller than optimum point SOR. Or, conversely, multigrid will yield a given accuracy in roughly one half the computing time of line SOR, or one third that of point SOR. These advantages increase with the number of nodes, and do not depend greatly on the various parameters. In particular, it is not necessary to find an optimum relaxation factor in the multigrid relaxation.

The choice of forcing functions to generate a coordinate system will depend on the particular problem and will necessarily involve some trial and error which must be repeated for every new problem. It is felt that in such applications, the present method could be used advantageously.

References

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